The topology of local commensurability graphs

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Abstract

We initiate the study of the p-local commensurability graph of a group, where p is a prime. This graph has vertices consisting of all finite-index subgroups of a group, where an edge is drawn between A and B if $[A : A\cap B]$ and $[B : A\cap B]$ are both powers of p. We show that any component of the p-local commensurability graph of a group with all nilpotent finite quotients is complete. Further, this topological criterion characterizes such groups. In contrast to this result, we show that for any prime p the p-local commensurability graph of any large group (e.g. a nonabelian free group or a surface group of genus two or more or, more generally, any virtually special group) has geodesics of arbitrarily long length.

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Let G be a group and p a prime number. Recall that two subgroups $\Delta_1 \leq G$ and $\Delta_2 \leq G$ are *commensurable* if $\Delta_1 \cap \Delta_2$ is finite-index in both Δ_1 and Δ_1. We define the *p-local commensurability graph* of G to be the graph with vertices consisting of finite-index subgroups of G where two subgroups $A, B \leq G$ are adjacent if and only if $[A : A\cap B][B : A\cap B]$ is a power of p. We denote this graph by $\Gamma_p(G)$. For a warm-up example, see Figure 1.

The goal of this paper is to draw algebraic information of G from the topology of $\Gamma_p(G)$.

Theorem 1. *Let G be a group. The following are equivalent:*

1. *For any prime p, every component of $\Gamma_p(G)$ is complete.*
2. *All of the finite quotients of G are nilpotent.*

The proof of Theorem 1 is in §2. The classification of finite simple groups and the structure theory of solvable groups play important roles in our proofs. Theorem 1 applies, for example, to Grigorchuk's group [Gri83], which is a 2-group and therefore has only nilpotent finite quotients.

In contrast to the above theorem, we show that components of the local commensurability graphs of free groups are far from complete:

Theorem 2. *Let F be a rank two free group. For any prime p and $N > 0$, there exist infinitely many geodesics γ, each in a different component of $\Gamma_p(F)$, such that the length of each γ is greater than N.*

We prove Theorem 2 in §3. A result of Robert Guralnick (which uses the classification of finite simple groups) concerning subgroups of prime power index in a nonabelian finite simple group is used in an essential way in our proof [Gur83]. Moreover, in our proof we get a clean description of an entire component of the p-local commensurability graph of many finite alternating groups. See Figure 2, for example.

Our next result demonstrates that arbitrarily long geodesics in the p-local commensurability graph of a free group cannot possibly all come from a single component. We prove this at the end of §1.

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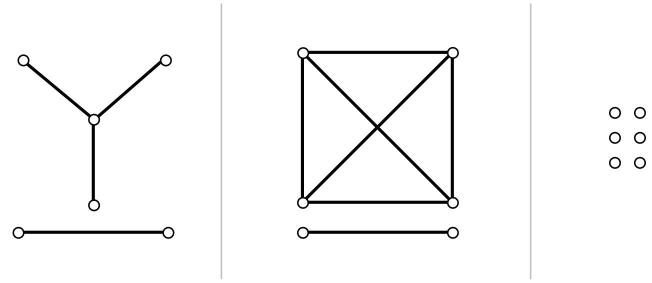

Figure 1: Let Sym_3 be the symmetric group on 3 elements (note Sym_3 is solvable and not nilpotent). The figure above displays $\Gamma_2(\mathrm{Sym}_3), \Gamma_3(\mathrm{Sym}_3)$, and $\Gamma_5(\mathrm{Sym}_3)$ in that order. All $\Gamma_p(\mathrm{Sym}_3)$ for primes $p > 3$ are discrete spaces.

Proposition 3. *Let G be any group. Let Ω be a connected component of $\Gamma_p(G)$. Then there exists $C > 0$ such that any two points in Ω are connected by a path of length less than C. That is, the diameter of Ω is finite. Moreover if any vertex of Ω is a normal subgroup of G then the diameter of Ω is bounded above by 3.*

As a consequence of Theorem 2 and Proposition 3, there exists components of the p-local commensurability graph of a nonabelian free group with no normal subgroups as vertices (see Corollary 22 at the end of §3).

Recall that a group is *large* if it contains a finite-index subgroup that admits a surjective homomorphism onto a non-cyclic free group. Such groups enjoy the conclusion of Theorem 2. See the end of §3 for the proof.

Corollary 4. *Let G be a large group. For any prime p and $N > 0$, there exists infinitely many geodesics γ, each in a different component of $\Gamma_p(G)$, such that the length of each γ is greater than N.*

Experiments that led us to the above theorems were done using GAP [GAP15] and Mathematica [Res15].

This paper sits in the broader program of studying infinite groups through their residual properties, which is an area of much activity (see, for instance, [KT], [BRK12], [BRM11], [GK], [BRHP], [BRS], [KM11], [Riv12], [Pat13], [LS03]). Specifically, a similar object is studied in the recent article [AAH$^+$15]. There a graph is constructed with vertices consisting of subgroups of finite index, and an edge is drawn between two vertices if one is a prime-index subgroup (the prime is not fixed) of the other. They show that for every group G, their graph is bipartite with girth contained in the set $\{4, \infty\}$ and if G is a finite solvable group, then their graph is connected.

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1 Preliminaries and basic facts

In this section we record some basic facts that will be used throughout. We start with a couple of elementary results.

Lemma 5. *Let $\pi : G \to G/N$ be a quotient map. For subgroups $K \le H \le G$ we have*

$$[H : K] = [\pi(H) : \pi(K)][H \cap N : K \cap N].$$

Proof. We know that

$$[H:K][K:K\cap N]=[H:K\cap N] \qquad \text{and} \qquad [H:H\cap N][H\cap N:K\cap N]=[H:K\cap N].$$

Equating left hand sides and rearranging terms yields

$$\frac{[H:H\cap N]}{[K:K\cap N]}=\frac{[H:K]}{[H\cap N:K\cap N]}.$$

Because $\pi(K)=KN/N=K/(K\cap N)$, and similarly for H, we see that

$$[\pi(H):\pi(K)]=\frac{|\pi(H)|}{|\pi(K)|}=\frac{[H:H\cap N]}{[K:K\cap N]}.$$

The desired result follows. ☐

Lemma 6. *Let N be a normal subgroup of G and p a prime. If A and N are both subgroups of index a power of p in G, then $[G:A\cap N]$ is also a power of p.*

Proof. Let $\pi: G\to G/N$ be the quotient map. Then $[A:A\cap N]=|\pi(A)|$. Because G/N is a p-group, it follows that $[A:A\cap N]$ is a power of p. Therefore $[G:A\cap N]=[G:A][A:A\cap N]$ is a power of p. ☐

Our next couple of lemmas give control of local commensurability graphs under some maps.

Lemma 7. *If G is a group, $\pi: G\to Q$ is a surjection, and γ a path in $\Gamma_p(G)$, then $\pi(\gamma)$ is a path in $\Gamma_q(Q)$ with length bounded above by the length of γ.*

Proof. If $K\le H\le G$ then $[\pi(H):\pi(K)]$ divides $[H:K]$ by Lemma 5. Therefore adjacent vertices in γ map to adjacent vertices in $\pi(\gamma)$, or are possibly identified in $\Gamma_p(Q)$. ☐

Lemma 8. *Suppose G is a group and p is prime.*

1. *If N is a normal subgroup of G, then the quotient map $\pi: G\to G/N$ induces an isometric graph embedding $\Gamma_p(G/N)\to\Gamma_p(G)$ as an induced subgraph.*
2. *If H is a finite-index subgroup of G, then the inclusion $i: H\to G$ induces a graph embedding $\Gamma_p(H)\to\Gamma_p(G)$ as an induced subgraph.*
3. *If N is a finite-index normal subgroup of G, then the inclusion $i: N\to G$ induces an isometric graph embedding $\Gamma_p(N)\to\Gamma_p(G)$ as an induced subgraph.*

Proof. For 1, if $\pi: G\to G/N$ is a quotient map, then the assignment $K\mapsto\pi^{-1}(K)$ defines a graph embedding $\Gamma_p(G/N)\to\Gamma_p(G)$ whose image is an induced subgraph. This embedding is isometric by Lemma 7.

For 2, if $H\le G$ has finite-index, then the assignment $K\mapsto i(K)$ defines a graph embedding $\Gamma_p(H)\to\Gamma_p(G)$ whose image is an induced subgraph.

For 3, let $N\lhd G$ be a finite-index subgroup, with assignment $\phi: K\mapsto i(K)$ defined over all subgroups K in N. Let $H_1,H_2\in\phi(\Gamma_p(N))$ and let $H_1=J_1,\dots,J_n=H_2$ be a path in $\Gamma_p(G)$ from H_1 to H_2. Then for each $i=1,\dots,n-1$, we have that

$$[J_i:J_i\cap J_{i+1}][J_{i+1}:J_i\cap J_{i+1}]$$

is a power of p. By Lemma 7, $\pi(J_1),\dots,\pi(J_n)$ is a path in $\Gamma_p(G/N)$. Because $J_1\le N$, this is a path of p-subgroups of G/N. Therefore $[J_i:J_i\cap N]$ is a power of p for all $i=1,\dots,n$. Thus, by Lemma 6 applied to $J_i\cap N$ and $J_{i+1}\cap J_i$, we have for $i=1,\dots,n-1$,

$$[J_i:(J_i\cap N)\cap(J_i\cap J_{i+1})][J_{i+1}:(J_{i+1}\cap N)\cap(J_i\cap J_{i+1})],$$

is a power of p. Hence, for $i = 1, \ldots, n-1$,

$$[J_i : N \cap J_i][N \cap J_i : N \cap J_i \cap J_{i+1}] = [J_i : N \cap J_i \cap J_{i+1}]$$

is a power of p giving that $[N \cap J_i : N \cap J_i \cap J_{i+1}]$ is a power of p, since above we showed that $[J_i : N \cap J_i]$ is a power of p. By a similar argument, we get that $[N \cap J_{i+1} : N \cap J_i \cap J_{i+1}]$ is a power of p, and thus $N \cap J_i$ and $N \cap J_{i+1}$ are adjacent in $\Gamma_p(G)$. It follows that the path $J_1, \ldots, J_n$ can be replaced by the path (which possibly has repeated vertices) $J_1 = J_1 \cap N, J_2 \cap N, \ldots, J_{n-1} \cap N, J_n \cap N = J_n$, which is entirely contained in $\Gamma_p(H)$. It follows that $\Gamma_p(H)$ is a geodesic metric space in the path metric induced from $\Gamma_p(G)$, as desired. □

Note that the hypothesis of normality in 3 cannot be removed. For example, suppose S and T are disjoint sets with $|S| = |T| = 5$ and consider the non-normal subgroup $\mathrm{Alt}_S \times \mathrm{Alt}_T \leq \mathrm{Alt}_{S \cup T}$. It can be shown using Lemma 18 below that Alt_S and Alt_T are in the same component of $\Gamma_5(\mathrm{Alt}_{S \cup T})$ but in different components of $\Gamma_5(\mathrm{Alt}_S \times \mathrm{Alt}_T)$.

Our next lemma will lead us to proving our first result concerning free groups.

Lemma 9. *Let A be a vertex in $\Gamma_p(G)$. Suppose B shares an edge with A. If q^k divides $[G : A]$ for some prime $q \neq p$ then q^k divides $[G : B]$.*

Proof. In this case, we have

$$[G : A \cap B] = [G : A][A : A \cap B] = [G : B][B : A \cap B].$$

Hence, if q^k divides $[G : A]$, then q^k must divide $[G : B]$ because $[B : A \cap B]$ is a power of p. □

Proposition 10. *The p-commensurability graph of a free group has infinitely many components.*

Proof. Any free group has subgroups $N_1, N_2, \ldots$ with distinct prime indices $q_1, q_2, \ldots$. By the previous lemma, any vertex that is in the connected component of N_i has index divisible by q_i. Thus, no path exists between N_i and N_j for distinct i, j. □

We finish this section by proving a general result: for any group G, any component of $\Gamma_p(G)$ has finite diameter.

Proof of Proposition 3. Let G be any group and Ω a component of $\Gamma_p(G)$. Take any vertex A in Ω and let N be the normal core of A. Let $\pi : G \to G/N$ be the quotient map. Let $D = \{BN : B \in \Omega\}$. We claim that the diameter of $\Gamma_p(G)$ is less than $|D| + 2$.

Let B be a subgroup in Ω. Let $V_1, \ldots, V_m$ be a path in $\Gamma_p(G)$ connecting A to B. Then by Lemma 5, $\pi(V_1), \ldots, \pi(V_m)$ is a path in $\Gamma_p(G/N)$ connecting $\pi(A)$ to $\pi(B)$. Hence

$$\pi(V_1)N, \cdots, \pi(V_m)N$$

is a path connecting A to BN, and so BN is an element of Ω. Further, if $[G : B] = np^r$ where $gcd(n, p^r) = 1$, then $[G : BN] = np^e$ by Lemma 9. Since $B \leq BN$ and $[G : BN][BN : B] = [G : B]$, we get

$$np^e[BN : B] = np^k$$

and therefore $[BN : B] = p^{k-e}$. Hence BN and B are adjacent in $\Gamma_p(G)$. It follows that there is an edge from any element in Ω to one in D, and so the diameter of Ω is bounded above by the diameter of the subgraph induced by D plus 2. This gives the desired bound $|D| + 2$.

If Ω contains a normal subgroup as a vertex then we can pick $A = N$ in the above argument. Therefore D is the set of p-subgroups of G/N. Any two such subgroups are connected by an edge, so the diameter of Ω is bounded above by 3. □

2 Nilpotent groups: The Proof of Theorem 1

We will prove Theorem 1 in two steps, as Propositions 12 and 15 below. For a finite nilpotent group G let $S_p(G)$ denote the unique Sylow p-subgroup of G. Recall that G is the direct product of its Sylow subgroups.

Lemma 11. *Suppose $G = S_{p_1}(G) \times \cdots \times S_{p_k}(G)$ for primes $p_1, \cdots, p_k$. Let $\pi_i : G \to S_{p_i}(G)$ be the quotient map for each i. Then any subgroup $H \leq G$ has the form $H = \pi_1(H) \times \cdots \times \pi_k(H)$.*

Proof. Choose $\ell_1, \dots, \ell_k$ so that $g^{p_i^{\ell_i}} = 1$ for all $g \in S_{p_i}(G)$. Choose N so that

$$N p_1^{\ell_1} \cdots p_{k-1}^{\ell_{k-1}} \equiv 1 \pmod{p_k^{\ell_k}}.$$

Take any $h \in H$ and write $h = (h_1, \dots, h_k)$ for $h_i \in S_{p_i}(G)$ for all i. Then

$$h^{N p_1^{\ell_1} \cdots p_{k-1}^{\ell_{k-1}}} = (1, \dots, 1, h_k).$$

Therefore $(1, \dots, 1, h_k) \in H$, and so we may identify $\pi_k(H)$ with a subgroup of H. Applying this argument to each other factor, the result follows. □

Proposition 12. *If G is a finitely generated group such that every finite quotient of G is nilpotent, then every component of $\Gamma_p(G)$ is complete for all p.*

Proof. Suppose A and B are subgroups of G in the same component of $\Gamma_p(G)$ for some prime p and take any path $A = P_0, P_1, \dots, P_n = B$ from A to B. Let N be a normal, finite-index subgroup of G contained in P_i for every i. Then G/N is a nilpotent group and $\pi(P_0), \pi(P_1), \dots, \pi(P_n)$ is a path in $\Gamma_p(G/N)$, where $\pi : G \to G/N$ is the quotient map.

Let $\mathcal{P}$ be a finite set of primes so that $G/N = \prod_{q \in \mathcal{P}} S_q(G/N)$. By Lemma 11 we have decompositions $\pi(P_i) = \prod_{q \in \mathcal{P}} S_q(\pi(P_i))$ for each i. It is straightforward to see that

$$\pi(P_i) \cap \pi(P_{i+1}) = \prod_{q \in \mathcal{P}} S_q(\pi(P_i)) \cap S_q(\pi(P_{i+1}))$$

for any i, and so for $j = i$ or $j = i + 1$ we have

$$[\pi(P_j) : \pi(P_i) \cap \pi(P_{i+1})] = \prod_{q \in \mathcal{P}} [S_q(\pi(P_j)) : S_q(\pi(P_i)) \cap S_q(\pi(P_{i+1}))].$$

Since $\pi(P_i)$ and $\pi(P_{i+1})$ are adjacent in the p-local commensurability graph of G/N, it follows that $S_q(\pi(P_i)) = S_q(\pi(P_{i+1}))$ for all i and all $q \neq p$. Therefore $S_q(\pi(A)) = S_q(\pi(B))$ for all $q \neq p$, and so $[\pi(A) : \pi(A) \cap \pi(B)][\pi(B) : \pi(A) \cap \pi(B)]$ is a power of p. Because $[K : L] = [\pi(K) : \pi(L)]$ for any subgroups $L \leq K \leq G$ containing N, this shows that A and B are adjacent in $\Gamma_p(G)$. □

Lemma 13. *If Q is a finite solvable group that is not nilpotent then there is some prime p so that a connected component of $\Gamma_p(Q)$ is not complete.*

Proof. Let Π be the set of prime divisors of the order of the finite solvable group Q. For any prime $q \in \Pi$ there is a Hall subgroup H_q so that $[Q : H_q] = q^k$ for some k and q does not divide the order of H_q. Because Q is not nilpotent, there is some prime p and a Hall subgroup H_p so that $g^{-1} H_p g \neq H_p$ for some $g \in Q$. Then both H_p and $g^{-1} H_p g$ are adjacent to Q in $\Gamma_p(Q)$, but there is no edge between H_p and $g^{-1} H_p g$ in $\Gamma_p(Q)$. □

Lemma 14. *If Q is a non-abelian finite simple group then Q contains a non-nilpotent solvable subgroup.*

Proof. By Theorem 1 in [BW97] any non-abelian finite simple group contains a *minimal simple group*, a non-abelian simple group all of whose proper subgroups are solvable. By the Main Theorem of [Tho73], minimal simple groups come from the following list:

$$PSL_2(q),\ Sz(q),\ PSL_3(3),\ M_{11},\ {}^2F_4(2), \text{ and } \mathrm{Alt}_7.$$

We show that each of these subgroups contains a solvable and non-nilpotent subgroup:

1. The groups $PSL_2(2) \cong SL_2(2) \cong \mathrm{Sym}_3$ and $PSL_2(3) \cong \mathrm{Alt}_4$ are non-nilpotent and solvable. Suppose now that $q-1 > 2$. As the group of units in a finite field of order q form a cyclic subgroup of order $q-1$, there exists a such that $a^2 \neq 1$. Hence, there exists a, b in any finite field of order q such that $ab = 1$ and $a \neq b$. The group in $SL_2(q)$ generated by
$$A := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
is solvable. Moreover, the order of A divides $(q-1)$ and the order of B divides q. Moreover, we have
$$ABA^{-1}B^{-1} = \begin{pmatrix} 1 & a/b-1 \\ 0 & 1 \end{pmatrix},$$
which is not central (the only central elements in $SL_2(q)$ are diagonal matrices). Hence, $\langle A, B\rangle$ has non-nilpotent image in $PSL_2(q)$, as elements of coprime order in a finite nilpotent group must commute.

2. Any Suzuki group by [Suz60, 4.] has a cyclic subgroup that is maximal nilpotent and of index 4 in its normalizer. Because every group of order 4 is nilpotent, this normalizer is solvable. Since the cyclic subgroup is maximal nilpotent, this normalizer cannot be nilpotent.

3. $PSL_3(3) \cong SL_3(3)$ contains $SL_2(3)$, which is solvable and not nilpotent.

4. M_{11} contains Sym_5 as a maximal subgroup by [Hun80] and hence contains Sym_3.

5. ${}^2F_4(2)$ contains $\mathrm{PSL}_2(25)$ as a maximal subgroup by [Hun80] and hence contains a subgroup that is solvable and not nilpotent by the above.

6. Alt_7 contains Alt_4.

□

Proposition 15. *Suppose G is a finitely generated group with a finite-index, normal subgroup N such that G/N is not nilpotent. Then there is some p so that a component of $\Gamma_p(G)$ is not complete.*

Proof. Take G and N as above, let $Q = G/N$ and let $\pi : G \to Q$ be the quotient map. If Q is solvable, then by Lemma 13 there is a prime p and subgroups $A, B \leq Q$ in the same component of $\Gamma_p(Q)$ that are not adjacent. Then $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are non-adjacent vertices in the same component of $\Gamma_p(G)$ by Lemma 8, so $\Gamma_p(G)$ is not complete.

Now consider the case that Q is not solvable. Let $Q = N_0 \geq N_1 \geq \cdots \geq N_{k-1} \geq N_k = \{1\}$ be any chain of subgroups such that N_{i+1} is a maximal normal subgroup of N_i for all i. Because Q is not solvable, there is some j so that N_j/N_{j+1} is not abelian. Then by Lemma 14 there is a non-nilpotent solvable subgroup $S \leq N_j/N_{j+1}$. By Lemma 13 there is some prime p with a component Ω of $\Gamma_p(S)$ that is not complete. By Lemma 8 the component Ω fully embeds in a component of $\Gamma_p(G)$, which is therefore not complete. □

3 Free groups: The Proof of Theorem 2

Let F be the free group of rank two. Let p be a prime and $N \in \mathbb{N}$ be given. By Lemma 8, to prove Theorem 2 it suffices to find a finite quotient Q of F with subgroups $A, B \leq Q$ such that the length of any geodesic in $\Gamma_p(Q)$ connecting A to B is greater than N. Our candidate for Q is Alt_X, the alternating group on a set X of more than $p^k > N$ elements, and our candidates for A and B are conjugates of Alt_S for a subset $S \subseteq X$ with p^k elements.

We first need a couple technical group theoretic results. First, we give a description of a connected component in $\Gamma_p(\mathrm{Alt}_X)$. This requires a simple lemma.

Lemma 16. *If $T_1 \cap T_2$ has more than one element and $|T_1|, |T_2| \geq 4$, then $\langle Alt_{T_1}, Alt_{T_2} \rangle = Alt_{T_1 \cup T_2}$.*

Proof. We prove this by induction on $|T_1 \cup T_2|$. The case that $T_1 = T_2$ is clear, so suppose $T_1 \neq T_2$. The base case, when $|T_1| = |T_2| = 4$ and $|T_1 \cap T_2| \in \{2, 3\}$, follows by computation (we did this in [GAP15]). For the inductive step, suppose without loss of generality that $x \in T_1 \setminus T_2$. By inductive hypothesis $\langle \mathrm{Alt}_{T_1 \setminus \{x\}}, \mathrm{Alt}_{T_2} \rangle = \mathrm{Alt}_{T_1 \cup T_2 \setminus \{x\}}$. Arguing similarly if $T_2 \setminus T_1$ is nonempty, we reduce to the case when $T_1 \cup T_2 \setminus T_1 \cap T_2$ consists of at most two points. To finish, we claim that any 3-cycle on points in $T_1 \cup T_2$ is in $\langle \mathrm{Alt}_{T_1}, \mathrm{Alt}_{T_2} \rangle$. Let v_1, v_2, v_3 be distinct points in $T_1 \cup T_2$. If $\{v_1, v_2, v_3\} \subseteq T_1$ or $\{v_1, v_2, v_3\} \subseteq T_2$, then we are done. Thus, by suitably relabeling, we may assume $v_1, v_2 \in T_1$ and $v_3 \in T_2$. Further, since $T_1 \cup T_2 \setminus T_1 \cap T_2$ consists of at most two points, then by relabeling again, we may assume $v_2 \in T_2$. Select $w_1, w_2 \in T_1 \cap T_2$ that are distinct from v_1, v_2, and v_3. Then, by the base case applied to $\mathrm{Alt}_{\{v_1, v_2, v_3, w_1\}} \leq \mathrm{Alt}_{T_1}$ and $\mathrm{Alt}_{\{v_1, v_2, v_3, w_2\}} \leq \mathrm{Alt}_{T_2}$, we obtain that $\mathrm{Alt}_{\{v_1, v_2, v_3, w_1, w_3\}}$ is contained in $\langle \mathrm{Alt}_{T_1}, \mathrm{Alt}_{T_2} \rangle$, and hence the desired 3-cycle is found. This completes the proof. $\square$

For any subset $S \subseteq X$, we denote the symmetric group on S by Sym_S and the alternating group on S by Alt_S. For a subgroup $P \leq \mathrm{Sym}_S$ we define the *support* to be the complement of the fixed point set of the action of P on S.

Lemma 17. *Let p be a prime number and k an integer so that $p^k > 4$. Let X be a finite set, $S \subseteq X$, and $P \leq Sym_X$ a p-group with support disjoint from S. Let E be an index p^j subgroup of $Alt_S \times P$. If $|S| = p^k$ or $|S| = p^k - 1$, then we have the decomposition $E = Alt_T \times P'$ for some $P' \leq P$ and some $T \subseteq S$ with $|T| = p^k$ or $|T| = p^k - 1$.*

Proof. Let $\pi : \mathrm{Alt}_S \times P \to \mathrm{Alt}_S$ be the projection map. By Lemma 5 we have

$$[\mathrm{Alt}_S \times P : E] = [\mathrm{Alt}_S : \pi(E)][1 \times P : E \cap (1 \times P)].$$

The left hand side of this equation is a power of p, so $[\mathrm{Alt}_S : \pi(E)]$ is a power of p. Because $|S| = p^k$ or $|S| = p^k - 1$ by assumption, Theorem 1(a) in [Gur83] immediately implies that either $\pi(E) = \mathrm{Alt}_S$ or $|S| = p^k$ and $\pi(E) = \mathrm{Alt}_{S \setminus \{v\}}$ for some $v \in S$. Let T denote the set such that $\pi(E) = \mathrm{Alt}_T$. Let q be 3 if $p \neq 3$ and q be 2 if $p = 3$. For the case $p \neq 3$, recall that Alt_T is generated by 3-cycles by elementary properties of alternating groups. In the case $p = 3$, note that $p^k > 6$. Because Alt_6 is generated by an element of order 2 and one of order 4, Lemma 16 implies that Alt_T is generated by elements of order 2 or 4 in this case. Therefore in either case it follows that Alt_T is generated by elements $g_1, \ldots, g_k$ each with order dividing a power of q. Since π maps onto Alt_T, we have that for each $i = 1, \ldots, k$, there exists $v_i \in P$ such that $(g_i, v_i) \in E$. Since $v_i \in P$, we have that the order of v_i is coprime with g_i, hence as $q \neq p$, there exists ℓ such that

$$(g_i, v_i)^\ell = (g_i, 1).$$

It follows then that E contains all of $\mathrm{Alt}_T \times 1$, and hence $E = \mathrm{Alt}_T \times P'$ where $P' \leq P$, as desired. $\square$

Let $\Omega_{S,X}$ be the component of $\Gamma_p(\mathrm{Alt}_X)$ containing Alt_S, and let $B_{S,X}$ denote the set of subgroups in $\Omega_{S,X}$ isomorphic to Alt_T for some $|T| \in \{p^k, p^k - 1\}$. For odd primes p, we get the following description:

Lemma 18. *Let $S \subseteq X$ be a set of cardinality p^k for some odd prime p such that $p^k > 4$. Vertices of the component $\Omega_{S,X}$ in $\Gamma_p(Alt_X)$ consist of two classes of subgroups:*

> **Type 1.** *subgroups of the form $\langle Alt_T, P\rangle$, where $|T| = p^k$ and $P \le Alt_X$, and*
>
> **Type 2.** *subgroups of the form $\langle Alt_T, P\rangle$, where $|T| = p^k - 1$ and $P \le Alt_X$.*

In either case, the subgroup is $Alt_T \times P$, where P is a p-group with support in T^c. Moreover, for all primes p, if V is of Type 1 or Type 2, the set T is uniquely determined by V.

Proof. We first show uniqueness of T. This implies that Type 1 and Type 2 are disjoint classes. Let V be a vertex with distinct decompositions $\mathrm{Alt}_{T_i} \times P_i$ with $|T_i| > 3$ and p-group P_i with support in T_i^c for $i = 1,2$ such that $T_1 \neq T_2$. If $T_i \cap T_j$ is empty, then

$$[V : \mathrm{Alt}_{T_1} \times \mathrm{Alt}_{T_2} \times 1][\mathrm{Alt}_{T_1} \times \mathrm{Alt}_{T_2} \times 1 : \mathrm{Alt}_{T_1} \times 1] = [V : \mathrm{Alt}_{T_1} \times 1] = |P_1|,$$

and thus $[\mathrm{Alt}_{T_1} \times \mathrm{Alt}_{T_2} \times 1 : \mathrm{Alt}_{T_1} \times 1] = |\mathrm{Alt}_{T_2}|$ must be a power of p. But this is impossible as $|\mathrm{Alt}_{T_2}|$ is either $(p^k)!/2$ or $(p^k-1)!/2$ for $p^k > 4$. Thus, T_1 and T_2 overlap. If $T_1 \neq T_2$ then $\mathrm{Alt}_{T_1} \times 1$ cannot be normal because Alt_{T_2} acts transitively on T_2. But $\mathrm{Alt}_{T_1} \times 1$ is clearly normal in $\mathrm{Alt}_{T_1} \times P_1$, so this is a contradiction. Therefore $T_1 = T_2$.

Since elements in $B_{S,X}$ are of Type 1 or 2, it suffices to show that any E that is adjacent to an element of Type 1 or 2 must itself be of Type 1 or 2.

Let E be adjacent to $V = \mathrm{Alt}_T \times P$ where P is a p-group with support in T^c and $|T| = p^k$ or $|T| = p^k - 1$. Then $E \cap V$ is a subgroup of $\mathrm{Alt}_T \times P$ of index a power of p. By Lemma 17, $E \cap V = \mathrm{Alt}_T \times P'$ or $E \cap V = \mathrm{Alt}_{T\setminus\{v\}} \times P'$ where $P' \le P$ and $v \in T$. We will therefore assume without loss of generality that E contains $\mathrm{Alt}_T \times 1 = \mathrm{Alt}_T$ as a subgroup of p power index.

Suppose that E does not leave T invariant. Let $T_1, T_2, \cdots, T_k$ be the orbit of E acting on T and note that E contains Alt_{T_i} for each i. Suppose $T_i \cap T_{i+1}$ has fewer than two elements for some i. The group Alt_{T_i} contains $\mathrm{Alt}_{T_i \setminus T_i \cap T_{i+1}}$, which includes a permutation of order 2 since $|T_i| > 4$. Hence E contains $\mathrm{Alt}_{T_i} \times \mathbb{Z}/2\mathbb{Z} \ge \mathrm{Alt}_{T_i}$. This is impossible, as Alt_{T_i} is of index p^k in E for an odd prime p. We therefore know that $T_i \cap T_{i+1}$ has more than two elements for every i. Then by applying Lemma 16 we conclude that E contains $\mathrm{Alt}_{T_1 \cup T_2 \cup \cdots \cup T_k}$. Since E contains Alt_T as a subgroup of prime power index and $T_1 \cup \cdots \cup T_k \neq T_1$, it follows that $|T_1 \cup \cdots \cup T_k| = p^k$ and in fact E contains $\mathrm{Alt}_{T_1 \cup \cdots \cup T_k}$ as a subgroup of index p^ℓ for some ℓ.

We may therefore assume, after replacing T with $T_1 \cup \cdots \cup T_k$ if necessary, that E leaves T invariant. Then $E \le \mathrm{Sym}_T \times Q$ where Q is a group with support disjoint from T. Let $\pi : \mathrm{Sym}_T \times Q \to \mathrm{Sym}_T$ be the projection onto the first coordinate. By Lemma 5, $[\pi(E) : \mathrm{Alt}_T]$ divides $[E : \mathrm{Alt}_T]$ and hence is a power of p. It follows that $\pi(E) = \mathrm{Alt}_T$, as Alt_T is a maximal subgroup of Sym_T of index two. Further, since Alt_T is normal, we apply Lemma 5 to the map $\psi : \mathrm{Alt}_T \times Q \to Q$ to see that $|\psi(E)|$ is a power of p. Applying Lemma 17 we obtain the desired conclusion. □

The prime $p = 2$ requires relaxing the conclusion of Lemma 18, since any symmetric group on three or more elements contains an alternating group of index 2.

Lemma 19. *Let $S \subseteq X$ be a set of cardinality 2^k such that $k > 2$. Vertices of the component $\Omega_{S,X}$ in $\Gamma_p(Alt_X)$ is at least one of two types:*

> **Type 1'.** *subgroups V such that $Alt_T \times 1 \le V \le Sym_T \times P$, where $|T| = 2^k$ and $P \le Alt_X$, and*
>
> **Type 2'.** *subgroups V such that $Alt_T \times 1 \le V \le Sym_T \times P$, where $|T| = 2^k - 1$ and $P \le Alt_X$.*

In either case, P is a 2-group with support in T^c.

Proof. Since elements in $B_{S,X}$ are of Type 1' or 2', it suffices to show that any E that is adjacent to an element of one of the types must itself be of one of the types.

Let E be adjacent to some V with $\mathrm{Alt}_T \times 1 \leq V \leq \mathrm{Sym}_T \times P$ where P is a 2-group. Because V has index a power of 2 in $\mathrm{Sym}_T \times P$, we know that $E \cap V$ also has index a power of 2 in $\mathrm{Sym}_T \times P$. Since $\mathrm{Alt}_T \times P$ is a normal subgroup of $\mathrm{Sym}_T \times P$, we have by Lemma 6 that $(\mathrm{Alt}_T \times P) \cap E \cap V$ has index a power of 2 in $\mathrm{Sym}_T \times P$, and hence in $\mathrm{Alt}_T \times P$. By Lemma 17, $E \cap V \cap (\mathrm{Alt}_T \times P) = \mathrm{Alt}_T \times P'$ or $E \cap V \cap (\mathrm{Alt}_T \times P) = \mathrm{Alt}_{T\setminus\{v\}} \times P'$ where $P' \leq P$ and $v \in T$. We conclude that $\mathrm{Alt}_T \times 1$ or $\mathrm{Alt}_{T\setminus\{v\}} \times 1$ has index a power of 2 in $E \cap V \cap (\mathrm{Alt}_T \times P)$, and hence has index a power of 2 in E. We will therefore assume without loss of generality that E contains $\mathrm{Alt}_T \times 1 = \mathrm{Alt}_T$ as a subgroup with index a power of 2, where $|T| = 2^k$ or $|T| = 2^k - 1$.

Suppose that E does not leave T invariant. Let $T_1, T_2, \cdots, T_k$ be the orbit of E acting on T and note that E contains Alt_{T_i} for each i. Suppose $T_i \cap T_{i+1}$ has fewer than two elements for some i. The group Alt_{T_i} contains $\mathrm{Alt}_{T_i \setminus T_i \cap T_{i+1}}$, which includes a permutation of order 3 because $|T_i| > 3$. Hence E contains $\mathrm{Alt}_{T_i} \times \mathbb{Z}/3\mathbb{Z} \geq \mathrm{Alt}_{T_i}$. This is impossible, as Alt_{T_i} is of 2 power index in E. We therefore know that $T_i \cap T_{i+1}$ has more than two elements for every i. Then by applying Lemma 16 we conclude that E contains $\mathrm{Alt}_{T_1 \cup T_2 \cup \cdots \cup T_k}$. Since E contains Alt_T as a subgroup of prime power index and $T_1 \cup \cdots \cup T_k \neq T_1$, it follows that $|T_1 \cup \cdots \cup T_k| = 2^k$ and in fact E contains $\mathrm{Alt}_{T_1 \cup \cdots \cup T_k}$ as a subgroup of index 2^ℓ for some ℓ.

We may therefore assume, after replacing T with $T_1 \cup \cdots \cup T_k$ if necessary, that E leaves T invariant. Then $\mathrm{Alt}_T \times 1 \leq E \leq \mathrm{Sym}_T \times Q$ where Q is a 2-group with support disjoint from T, as desired. □

Note that groups of Type 1 and Type 2 are of Type 1' and Type 2' respectively. The next result allows us to restrict attention to geodesics in $B_{S,X}$ when computing distances there.

Lemma 20. *Let $S \subseteq X$ be a set of cardinality $p^k > 4$ for some prime p and integer k. Then $B_{S,X}$ is a geodesic metric space in the path metric induced from $\Omega_{S,X}$.*

Proof. We first need a local fact. Let V_1, V_2 be two adjacent vectors in $\Omega_{S,X}$. If p is odd, then by Lemma 18 we have $V_i = \mathrm{Alt}_{T_i} \times P_i$, where $|T_i| = p^k$ or $|T_i| = p^k - 1$ and the support of P_i is disjoint from T_i for $i = 1,2$. If $p = 2$, then by Lemma 19, $\mathrm{Alt}_{T_i} \times 1 \leq V_i \leq \mathrm{Sym}_{T_i} \times P_i$, where $|T_i| = p^k$ or $|T_i| = p^k - 1$ and the support of P_i is disjoint from T_i for $i = 1,2$. We claim that in either case, Alt_{T_1} and Alt_{T_2} are connected by an edge in $\Gamma_p(\mathrm{Alt}_X)$.

Since V_1 and V_2 are adjacent, we have that

$$[V_1 : V_1 \cap V_2][V_2 : V_1 \cap V_2]$$

is a power of p. Thus, when p is odd, Lemma 17 applied twice along with the uniqueness in Lemma 18 gives that $V_1 \cap V_2$ is $\mathrm{Alt}_S \times P$ where $S \subseteq T_1 \cap T_2$ satisfies $|S| = p^k$ or $|S| = p^k - 1$ and $P \leq P_1 \cap P_2$. Thus it is straightforward to see that Alt_{T_1} is adjacent to Alt_{T_2}.

When $p = 2$, set $H_i = \mathrm{Alt}_{T_i} \times 1$ and $\Lambda = V_1 \cap V_2$. Then H_i is normal in V_i, thus $H_i \cap \Lambda$ is normal in Λ. Since $[\mathrm{Sym}_{T_i} \times P_i : H_i]$ is a power of 2 and

$$[\mathrm{Sym}_{T_i} \times P_i : V_i][V_i : H_i] = [\mathrm{Sym}_{T_i} \times P_i : H_i],$$

we get $[V_i : H_i]$ is a power of 2. Since H_i is normal in V_i, Lemma 6 implies that $[V_i : H_i \cap \Lambda]$ is a power of 2. Further, as $[V_i : \Lambda]$ is a power of 2 and

$$[V_i : \Lambda][\Lambda : H_i \cap \Lambda] = [V_i : H_i \cap \Lambda]$$

we conclude that $[\Lambda : H_i \cap \Lambda]$ is a power of 2 for $i = 1,2$. Thus, applying Lemma 6 to $H_1 \cap \Lambda \lhd \Lambda$ and $H_2 \cap \Lambda \lhd \Lambda$, we have that $H_1 \cap H_2 \cap \Lambda$ has index a power of 2 in Λ. As

$$[V_i : \Lambda][\Lambda : H_1 \cap H_2 \cap \Lambda] = [V_i : H_1 \cap H_2 \cap \Lambda],$$

it follows that $[V_i : H_1 \cap H_2 \cap \Lambda]$ is a power of 2. Because $[V_i : H_i]$ is also a power of 2 (shown above) and

$$[V_i : H_i][H_i : H_1 \cap H_2 \cap \Lambda] = [V_i : H_1 \cap H_2 \cap \Lambda]$$

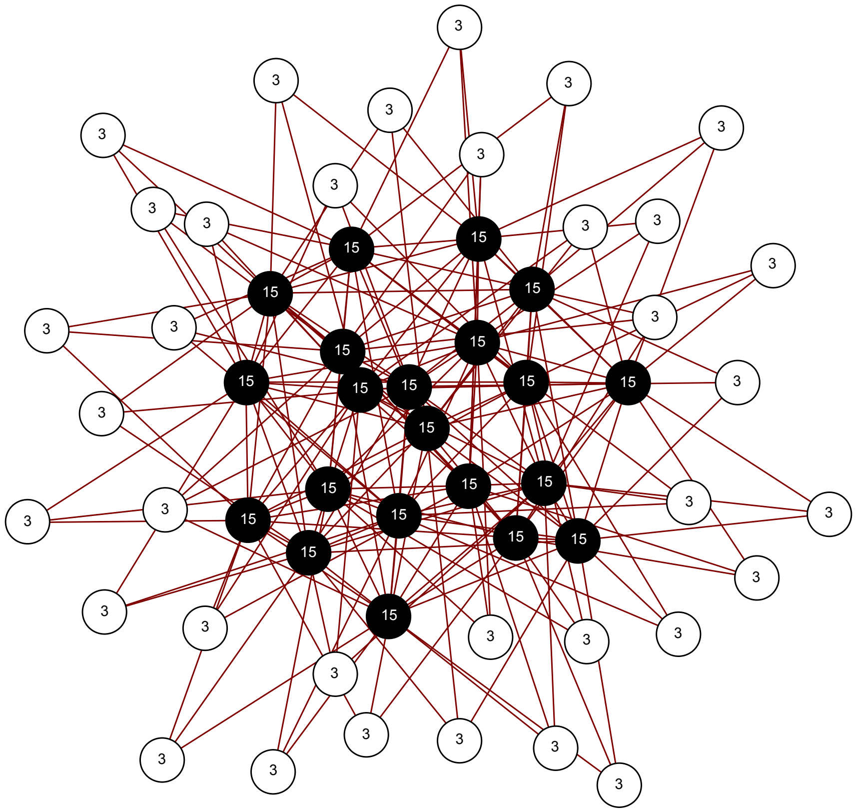

Figure 2: $\Omega_{S,X}$ with $|S| = 5$ and $|X| = 7$. The coloring gives the types and the numbers give the valence of each vertex. This figure was generated using GAP [GAP15] and Mathematica [Res15]

we have $[H_i : H_1 \cap H_2 \cap \Lambda]$ is a power of 2 for each i. By applying Theorem 1(a) in [Gur83] and the uniqueness in Lemma 18, we have $H_1 \cap H_2 \cap \Lambda$ is Alt_S for some $S \subseteq T_1 \cap T_2$ with $|S| = p^k$ or $|S| = p^k - 1$. Thus, Alt_{T_1} is adjacent to Alt_{T_2}, as claimed.

Now let γ be a path in $\Omega_{S,X}$ that, except for its endpoints, is entirely in the complement of $B_{S,X}$. Enumerate the vertices of γ in the order they are traversed,

$$V_1, V_2, \ldots, V_m, \text{ where } \mathrm{Alt}_{T_i} \times 1 \leq V_i \leq \mathrm{Sym}_{T_i} \times P_i \text{ for all } i = 1, \ldots, m$$

Then by the previous claim, we may form a new path (after throwing out repeated vertices)

$$\mathrm{Alt}_{T_{i_1}}, \mathrm{Alt}_{T_{i_2}}, \ldots, \mathrm{Alt}_{T_{i_n}}.$$

that is entirely contained in $B_{S,X}$ and has the same endpoints as γ. It follows that $B_{S,X}$ is geodesic in $\Omega_{S,X}$, as desired. □

Proposition 21. *Let $S \subseteq X$ be a set of cardinality $p^k > 4$ for some prime p and integer k. There exists $V, W \in B_{S,X}$ such that any path in $\Omega_{S,X}$ connecting V to W has length at least $p^k - \max\{0, 2p^k - |X|\}$.*

Proof. By Proposition 20, it suffices to show that there exists $V,W \in B_{S,X}$ such that any path in $B_{S,X}$ has length greater than $|X| - p^k$. Let $O_1, O_2 \subseteq X$ with $|O_1 \cap O_2| \le \max\{0, 2p^k - |X|\}$. Let $E_1, E_2, \ldots, E_m$ be distinct vertices in a non-back-tracking path in $B_{S,X}$ connecting Alt_{O_1} to Alt_{O_2}. Let $T_1, T_2, \ldots, T_m$ be subsets of X such that $E_i = \mathrm{Alt}_{T_i}$ for $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, we have one of three cases:

1. E_i is Type 1 and E_{i+1} is Type 1: In this case, $|T_{i+1} \cap T_i| = |T_i| - 1 = p^k - 1$.
2. E_i is Type 1 and E_{i+1} is Type 2: In this case, $T_{i+1} \subset T_i$ and $|T_{i+1}| = |T_i| - 1 = p^k - 1$.
3. E_i is Type 2 and E_{i+1} is Type 1: In this case, $T_{i+1} \supset T_i$ and $|T_{i+1}| = |T_i| + 1 = p^k$.
4. E_i is Type 2 and E_{i+1} is Type 2: This case never occurs, as $[\mathrm{Alt}_T : \mathrm{Alt}_U]$ is not a power of p for any proper subset $U \subset T$ with $|T| = p^k - 1$.

Thus, we see that for each i, we see that T_i and T_{i+1} differ by moving, adding, or removing at most one element. It follows that $m \ge p^k - |O_1 \cap O_2| \ge p^k - \max\{0, 2p^k - |X|\}$. □

Proof of Theorem 2. Let F be a rank two free group and p a prime. Given $N > 0$, choose k so that $p^k > N$ and $p^k > 4$. For any finite set X with $|X| > 2p^k$, let γ_X be a path of length p^k in $\Gamma_p(\mathrm{Alt}_X)$ guaranteed by Proposition 21. Then pulling back γ_X over any surjection $\pi : G \to \mathrm{Alt}_X$ produces a path of length p^k in $\Gamma_p(F)$ by Lemma 8. By Lemma 9, sets X_1 and X_2 with relatively prime cardinalities will produce geodesics in different components of $\Gamma_p(F)$. □

Proof of Corollary 4. Let G be a large group, p a prime, and $N > 0$. Since a finite-index subgroup of a nonabelian free group is nonabelian, there exists a normal finite-index subgroup $H \le G$ that surjects onto F, the free group of rank 2. By Lemma 7 and Theorem 2, there exists vertices $V,W \in \Gamma_p(H)$ such that any path connecting them in $\Gamma_p(G)$ has length greater than N. The result now follows from Lemma 8, as $\Gamma_p(H)$ isometrically embeds into $\Gamma_p(G)$. □

Corollary 22. *Let G be a large group and p be a prime. There exists a connected component of $\Gamma_p(G)$ that does not contain any normal subgroup.*

Proof. By Proposition 3, any component of $\Gamma_p(G)$ containing a normal subgroup as a vertex has diameter at most 3. By Corollary 4, there are components of G with arbitrarily long geodesics. □

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